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A generalized principal ideal theorem

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Abstract

The main theorem of this article is an extension of the generalized principal ideal theorem for ideals in Noetherian rings. Instead of requiring the rings to be Noetherian, some natural requirements are imposed on the chains of prime ideals under consideration. The standard (Noetherian) version of the generalized principal ideal theorem is deduced as a corollary and two other applications are presented.

MSC: 13C15

A cornerstone of the dimension theory of Noetherian rings is the famous principal ideal theorem of Krull, or more correctly, its generalization commonly referred to as the 'generalized principal ideal theorem'. It states that any minimal prime ideal of an ideal generated by n elements in a Noetherian ring has height at most n (e.g. see Theorem 18 of [5]). There is a long history of investigations attempting to further generalize this theorem to various classes of non-Noetherian rings. We do not attempt to present a detailed historical survey in this short article; instead the interested reader is urged to consult articles [1,2] and [4]. In the opposite direction, the failure of the generalized principal ideal theorem for non-Noetherian rings has also been exemplified, especially strikingly, in the series of examples presented in [3]. The main theorem of this article provides an extension of the generalized principal ideal theorem to rings which are not necessarily Noetherian. We have carefully avoided making any direct use of the generalized principal ideal theorem for Noetherian rings (which we subsequently recover in our first corollary). Thus our theorem can serve as a fitting replacement of the generalized principal ideal theorem. The corollaries demonstrate the utility of the main theorem as well as the relative advantage of our formulation of the generalized principal ideal theorem in comparison to its older versions.

In the following, a ring is tacitly assumed to be commutative with $1 \neq 0$. By a 'quasi-local ring' we mean a ring with only one maximal ideal. A quasi-local ring is called a 'local ring' only if it is Noetherian. As usual, if Q is a prime ideal of a ring and m is a positive integer, then by $Q^{(m)}$ we denote the m-th symbolic power of Q.

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Lemma. *Let R be a ring and let J be a finitely generated ideal of R. Then the following are equivalent.*

(i) *R/J^n is Noetherian for all positive integers n.*
(ii) *R/J^n is Noetherian for some positive integer n.*
(iii) *R/J is Noetherian.*

Proof. It suffices to derive assertion (i) from assertion (iii). Assume R/J is Noetherian. Then a prime ideal of R containing J is finitely generated as an ideal of R. Since a prime ideal of R contains J^n if and only if it contains J, we conclude that every prime ideal containing a power of J is a finitely generated ideal of R. Hence every prime ideal of R/J^n is finitely generated. $\square$

Theorem. *Let R be a ring with prime ideal M. Suppose for a positive integer n there are elements $a_1, \ldots, a_n$ of M such that M is a minimal prime of $Ra_1 + \cdots + Ra_n$. Assume there exists a chain*

$$M = P_1 \geq \cdots \geq P_t$$

of prime ideals of R such that

(i) *$t > n$,*
(ii) *$P_i \neq P_{i+1}$ for $1 \leq i < t$,*
(iii) *$P_t R_M$ is finitely generated and*
(iv) *$R_M/P_t R_M$ is Noetherian.*

Then

(i) *$t = n + 1$ and*
(ii) *there is a positive integer m such that*

$$(0 : P_t^m) \not\subseteq P_t.$$

In particular, every element of P_t is a zero-divisor of R and P_t is a minimal prime of 0.

Proof. Replacing R by its localization at M if necessary, henceforth we assume R to be a quasi-local ring with maximal ideal M. Then we have

$$\sqrt{Ra_1 + \cdots + Ra_n} = M.$$

If $P \supseteq P_t$ is a prime ideal of R, then from the hypotheses it follows that P is finitely generated and R/P is Noetherian. Let V denote the set of prime ideals of R containing P_2 and properly contained in M. Clearly, V is partially ordered by inclusion. Since M is the radical of a finitely generated ideal, V is inductive. Hence V has maximal elements. Replacing P_2 by a maximal element of V without any loss, we assume that there is no prime ideal of R strictly between $M = P_1$ and P_2. Our proof proceeds by induction on n.

The $n = 1$ case. In this case, since M is finitely generated and it is the radical of Ra_1, the factor ring $A := R/Ra_1$ is Artinian. Let $a := a_1$ and $P := P_2$. For a positive integer r let $I_r := P^{(r)}/Ra$. Then $\{I_r\}$ is a descending chain of ideals of the Artinian ring A and hence it is eventually stationary. Let m be a positive integer such that $I_r = I_m$ for all $r \geq m$. Then $P^{(r)} + Ra = P^{(m)} + Ra$ for all $r \geq m$. In particular, we have

$$P^{(r)} \subseteq P^{(m)} \subseteq P^{(r)} + Ra$$

for all $r \geq m$. Since $P \neq M$, the element a is not in P. Consequently, $(P^{(m)} : Ra) = P^{(m)}$. Let $I \subseteq P$ be a finitely generated ideal of R with R/I Noetherian (e.g. $I = P$). In view of the above lemma, R/I^r is Noetherian for all positive integers r. Fix a positive integer $r \geq m$. Since I^r is contained in $P^{(r)}$, the ring $R/P^{(r)}$ is also Noetherian. Let $B := R/P^{(r)}$, $J := P^{(m)}/P^{(r)}$ and let $b := a \mod P^{(r)}$. Then b is a nonunit of the local ring B, $J \subseteq Bb$ and $(J : Bb) = J$. It follows that $J = 0$. In other words, $P^{(r)} = P^{(m)}$ for all $r \geq m$. Consider the prime ideal $Q := P/I^{m+1}$ of the local ring $C := R/I^{m+1}$. Observe that $Q^{(j)} = P^{(j)}/I^{m+1}$ for all $j \geq 1$. Also, since C is Noetherian, we have

$$\bigcap_{j \geq 1} Q^{(j)} = \{z \in C \mid cz = 0 \text{ for some } c \in C \setminus Q\}.$$

Consequently

$$\bigcap_{j \geq 1} P^{(j)} = \{x \in R \mid sx \in I^{m+1} \text{ for some } s \in R \setminus P\}.$$

But the intersection on the left is just $P^{(m)}$, so we have

$$P^{(m)} = \{x \in R \mid sx \in I^{m+1} \text{ for some } s \in R \setminus P\}.$$

Since I is finitely generated, there exists an $s \in R \setminus P$ such that $sI^m \subseteq I^{m+1}$. Now a standard argument shows the existence of a positive integer k and an element $w \in I$ such that $s^k + w$ belongs to $(0 : I^m)$. Obviously $s^k + w$ is not in P. In particular, by taking $I = P$, we see that $(0 : P^m)$ is not contained in P. Observe that the canonical homomorphism from R to R_P maps P^m to 0. So the maximal ideal of R_P consists entirely of nilpotents. Consequently, P cannot properly contain any prime ideal of R. Thus $t = 2$ and our assertion is established.

The $n \geq 2$ case. Assume $n \geq 2$. Let $P := P_2$. Since $P \neq M$, at least one of $a_1, \ldots, a_n$ is not in P. Without loss we assume that $a := a_1$ is not in P. Now M being the only prime ideal strictly larger than P, we have $\sqrt{P + Ra} = M$. Let d be a positive integer such that $M^d \subseteq P + Ra$. Existence of such a d is assured by the earlier observation that M is finitely generated. Pick $c_2, \ldots, c_n$ in P and $b_2, \ldots, b_n$ in R such that $a_i^d = b_i a + c_i$ for $2 \leq i \leq n$. Then $\sqrt{Ra + Rc_2 + \cdots + Rc_n} = M$. Fix a positive integer e such that $M^e \subseteq Ra + Rc_2 + \cdots + Rc_n$.

Let $Q \subseteq P$ be a minimal prime of $Rc_2 + \cdots + Rc_n$ (this Q bears no relation to the Q appearing in the proof of the $n = 1$ case above). Let $\overline{R} := R/Q$ and $\overline{P} := P/Q$. Since $M^e \subseteq Q + Ra$, the maximal ideal $\overline{M} := M/Q$ of the quasi-local domain $\overline{R}$ is the radical of $\overline{R}\overline{a}$ where $\overline{a} := a \bmod Q$. Since a is not in P it is not in Q and hence $\overline{a} \neq 0$. Let $I \subseteq P_t$ be a finitely generated ideal with R/I Noetherian (e.g. $I = P_t$). Let $\overline{I} := I/Q$. Observe that $\overline{I}$ is a finitely generated ideal contained in $\overline{P}$ and $\overline{R}/\overline{I}$ is Noetherian. Also $\overline{M} \neq \overline{P}$. Applying the above established $n = 1$ case to the data $\overline{R}, \overline{M}, \overline{a}$ etc, we surmise that every element of $\overline{I}$ is a zero-divisor of the integral domain $\overline{R}$ and the prime chain $\overline{M} > \overline{P} \geq 0$ can have at most two distinct members. Hence $\overline{P} = 0$. It follows that P is a minimal prime of $Rc_2 + \cdots + Rc_n$.

Now consider the quasi-local ring $S := R_P$. The canonical images of $a, c_2, \ldots, c_n$ in S are denoted by the same symbols. Then, the maximal ideal of S is the radical of $Sc_2 + \cdots + Sc_n$ and $PS = P_2 S > \cdots > P_t S$ is a chain of $(t-1)$ distinct prime ideals of S with $(t-1) > (n-1)$. Clearly $P_t S$ is finitely generated and $S/P_t S$ is Noetherian. By the induction hypothesis $t - 1 = n$, there is a positive integer m and an element $f \in S \setminus P_t S$ such that $f(P_t S)^m = 0$. It is straightforward to derive the existence of a $g \in R \setminus P_t$ such that $gP_t^m = 0$, thereby establishing the desired assertion. □

As an immediate consequence, we have the well-known generalization of the (Krull's) principal ideal theorem.

Corollary 1. *If R is a Noetherian ring and I is an ideal of R generated by n elements, then the height of I is at most n.*

In [4], the authors consider rings R which are what they call r-Noetherian. We briefly recall the relevant definitions below (keeping the reader's convenience in mind) and then follow it with the second corollary of the theorem. It generalizes the first corollary by weakening the requirement on R from being Noetherian to being r-Noetherian.

Definitions. Let R be a ring. By a *regular* element of R we mean an (nonzero) element which is not a zero-divisor of R. An ideal of R is called an *r-ideal* if it contains a regular element of R. Likewise, a prime ideal which contains a regular element of R is called an r-prime ideal. If every r-ideal of R is finitely generated, R is called *r-Noetherian*. For a prime ideal P of R the *r-height* of P is defined to be 0 if P is not an r-ideal; otherwise, it is defined to be the largest t (either a positive integer or ∞) such that there exists a chain $P = P_1 > \cdots > P_t$ of distinct r-prime ideals of R. By the r-height of an ideal I of R we mean the minimum of the r-heights of the minimal primes of I.

Corollary 2. *Let R be an r-Noetherian ring. Suppose an r-ideal I of R can be generated by n elements. Then the r-height of any minimal prime of I does not exceed n.*

Remarks. 1. For each positive integer $n \geq 3$ there exists a quasi-local unique factorization domain R_n of dimension n such that the maximal ideal M_n of R_n is generated by two elements (see [3]). If $n \geq 4$, then there is a chain of

n distinct prime ideals $M_n := P_1 > \cdots > P_n$ of R_n such that P_n has height 1. Since R_n is a unique factorization domain, P_n is a principal ideal. Observe that only the hypothesis (iv) of the above theorem is not satisfied. Thus we cannot entertain any hope of bounding the parameter t relying solely on (i), (ii) and (iii). Nevertheless, it may be possible to replace (iv) by some carefully formulated assumption regarding the ideal-adic topologies of the factor domain $R_M/P_t R_M$.

2. It is easy to see that the hypothesis (iii) is also necessary. For example, let R be a valuation domain of a discrete valuation of (finite) rank $n \geq 2$. Then the maximal ideal M of R is a principal ideal. Consider the prime chain $M > P$ where P is the one-dimensional prime ideal of R. Our hypotheses (i), (ii) and (iv) are indeed valid in this case. Evidently, the second part of the theorem's conclusion fails to hold.

Finally, we present a sample of what additional information may be obtained from our theorem in comparison to its prior versions.

Corollary 3. *Let R be a quasi-local integral domain of dimension ≥ 2. Assume that the maximal ideal M of R is the radical of a principal ideal. Let P be a prime ideal of R with* $\dim R/P = 1$.

(i) *If R/P is Noetherian, then P is not finitely generated.*
(ii) *If M is finitely generated, then R does not have any one-dimensional finitely generated prime ideal.*

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